LOCALITY PROPERTIES OF THE PROBLEM OF HYDRODYNAMIC STABILITY

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Locality properties are formulated for short-wavelength disturbances in the problem of hydrodynamic stability, which together with global flow stability enable us to study the stability of particular sections of the stream, e.g., the flow core or the zone next to the wall. The locality properties are illustrated in the spectrum of small perturbations of plane Poiseuille flow and flows which are obtained by deforming a small section of the Poiseuille parabola. Such a deformation produces points of inflection which lead to the appearance of growing perturbations with wavelength of the order of the deformation zone. It is shown that discontinuities in the velocity profile leads to the loss of stability for high enough Reynolds' numbers.

1. Formulation of the Problem. The problem of hydrodynamic stability of the plane-parallel flow of a viscous incompressible fluid reduces to an analysis of the eigenvalue spectrum of the Orr–Summerfeld equation [1]

$$\varphi^{IV} - 2\alpha^2\varphi'' + \alpha^4\varphi = i\alpha R\,[(u - C)(\varphi'' - \alpha^2\varphi) - u''\varphi] \quad (-1 \leqslant y \leqslant 1) \tag{1.1}$$

Here u(y) is the velocity profile of the flow under investigation, $\varphi(y)$ is the complex amplitude of the stream function for a harmonic perturbation, α is the wavenumber, R is the Reynolds' number, $C = X + iY$ is the required eigenvalue, X is the phase velocity of the perturbation, and Y characterizes the development of the perturbation in time ($Y < 0$ corresponds to exponential damping).

A nontrivial solution of (1.1) should satisfy four homogeneous boundary conditions, for example, the condition of adherence to the walls:

$$\varphi(\pm 1) = \varphi'(\pm 1) = 0 \tag{1.2}$$

For sufficiently smooth u(y) and finite R a denumerable set of functions $C_n(\alpha)$, $n = 1, 2, 3, \ldots$, exists where $0 \leq \alpha < \infty$. However, in order to analyze the stability of any profile $u(y) \in C_2(-1, 1)$ for a fixed R it suffices to confine ourselves to investigating a finite number of spectral modes and a finite range of variation of α.

In fact we shall assume that for certain values of the parameters

$$|C| \gg \max(|u|, |u''|) \tag{1.3}$$

and so neglecting terms containing u and u" in (1.1), we have in the first approximation

$$C \approx -i(\beta_n^2 + \alpha^2)/\alpha R \tag{1.4}$$

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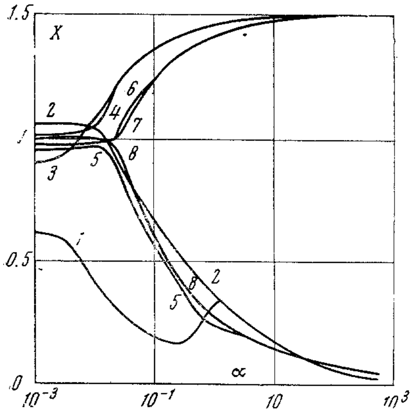

Fig. 1

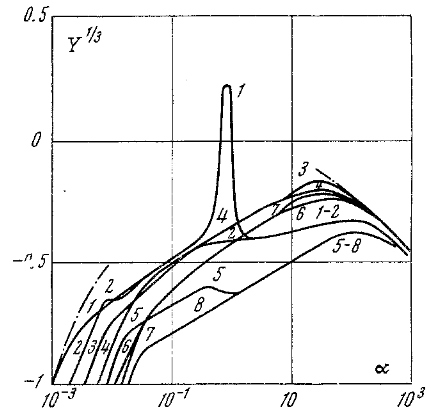

Fig. 2

where β_n are numbered in order of increasing modulus of roots of the equation

$$(\beta \operatorname{tg} \beta + \alpha \operatorname{th} \alpha)(\alpha \operatorname{tg} \beta - \beta \operatorname{th} \alpha) = 0$$
$$\pi n < 2 \mid \beta_n \mid \leqslant \pi (n + 1) \tag{1.5}$$

The initial assumption (1.3) is satisfied for arbitrary values of α if n is large enough, and for arbitrary n if α is small enough or large enough. Under these conditions (1.4) is valid, and thus Y is a large negative number independent of u(y). However, for small spectral numbers n and intermediate values of α the eigenvalue $C_n(\alpha)$ is strongly dependent on the form of the profile u(y).

If the following series expansion is made for small α,

$$C = C_0 / \alpha + C_1 + \alpha C_2 + \ldots \tag{1.6}$$

it is not difficult to see that terms in the series depend on the integral of u(y). From (1.4) the first term is independent of u(y) in general, while for the second term, for example, one can obtain, in the case of symmetric perturbations,

$$C_1 = \int_0^1 (1 + 2 \cos \pi n \cos \pi n y - 3 \cos 2\pi n y)\, u dy \tag{1.7}$$

etc. Thus, for small α the eigenvalues depend on integrals of the profile, i.e., long-wavelength perturbations are signs of the global average stability of the flow.

On the other hand, small local "roughnesses" of the velocity profile will not affect the behavior of the long-wavelength perturbations, and any instability possible under these conditions will be associated with the growth of short-wavelength perturbations whose wavelength is of the order of the roughness scale.

2. Locality Properties. Let

$$\sqrt{\alpha R} \gg 1, \qquad \alpha \gg 1 \tag{2.1}$$

Then the four fundamental solutions of (1.1) are described qualitatively by the following relations [1] outside a small region in the neighborhood of the critical point y_c (such that $u(y_c) = X$):

$$\varphi_{1,2} \sim \exp\left\{\pm \int_{y_c}^{y} \sqrt{i\alpha R (u - C)}\, dy\right\}, \quad \varphi_{3,4} \sim \exp\{\pm \alpha (y - y_c)\} \tag{2.2}$$

It should be noted that all the fundamental solutions satisfy the necessary condition of smoothness over the whole interval (−1, 1). Equation (2.2) only stresses the fact that four such fundamental solutions can be chosen, the modulus of whose amplitude will decrease or increase exponentially outside the small neighborhood of y_c.

For simplicity, we consider the case when y_c is sufficiently far away from the boundaries of the interval. We construct a linear combination φ_* of the two solutions of (2.2) whose amplitudes decrease in modulus as we move to the right of y_c, and a linear combination φ_{**} of the solutions which decrease to the left of y_c.

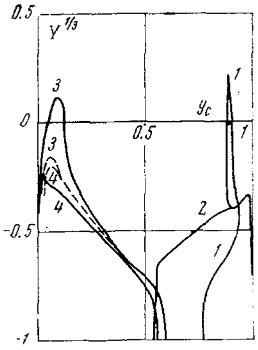

Fig. 3

Requiring that φ_* and φ_{**}, as well as their first three derivatives, should coincide at the point y_c (or close to it), we obtain the characteristic equation for C. The function φ_0 constructed in this way decreases exponentially on both sides of the critical point y_c, and if

$$k = \min\left\{|\alpha(y_c - y_1)|, \left|\int_{y_c}^{y_1} \sqrt{i\alpha R(u - C)}\,dy\right|\right\}$$

(where y_1 is one of the boundary points), the modulus of the amplitude of φ_0 close to the boundary points will be exp k times less than its value in the neighborhood of y_c, and in view of the conditions (2.1) φ_0 and its derivatives will be vanishingly small. Many arbitrary homogeneous boundary conditions will be satisfied with an accuracy of the order exp (−k). We now satisfy the boundary conditions exactly, by adding to φ_0 two fundamental solutions at each of the boundaries, which decrease rapidly as we move from the boundary towards y_c. In the region of y_c these additions will then be of the order exp (−2k).

Imposing the strict requirement that the solution should be continuous in the neighborhood of y_c, we find corrections to C and to the two solutions which decrease as we move away from the critical point, which will also be of the order exp (−2k). Continuing this process which converges rapidly in view of the conditions (2.1), we obtain a function φ in the limit which will be a nontrivial solution of (1.1) for homogeneous boundary conditions.

The eigenfunction constructed in this way is nonzero only in the small neighborhood of y_c (finite) for all practical purposes, and the dimension of the zone of finiteness, taking $R > \alpha$, is

$$|y - y_c| \lesssim 1/\alpha \tag{2.3}$$

In view of what has been said, the following locality properties can be formulated:

a) for practical purposes, the amplitude of the short-wave perturbation is nonzero only in the neighborhood (of diameter $\sim 1/\alpha$) of the point where the phase velocity of the perturbation coincides with the local velocity of the flow;

b) the nature of the homogeneous boundary conditions lying outside the finiteness zone does not affect the magnitude of the corresponding eigenvalue;

c) the magnitude of the eigenvalue depends only on the nature of the velocity profile of the fundamental flow in the finiteness zone (2.3). An arbitrary deformation of the velocity profile far from the critical point y_c does not affect the behavior of the short-wave perturbation.

3. The Eigenvalue Spectrum for Poiseuille Flow. A large number of papers, beginning with the investigation of Lin [1], have been devoted to the stability of plane Poiseuille flow:

$$u = {}^3/_2(1 - y^2) \qquad \left(\int_0^1 u\,dy = 1\right) \tag{3.1}$$

The spectrum of small perturbations for some fixed values of α as a function of the Reynolds' number has been calculated by Salwen and Grosch [2]. To illustrate the locality properties we shall consider the spectrum for a fixed Reynolds' number $R = 10^4$ over the whole range of variation of the wavenumber. Calculations carried out by the authors are in complete agreement with those of [1, 2], where they intersect.

Figures 1 and 2 present the functions $C_n(\alpha)$ for the first eight spectral numbers (altogether 12 were calculated). The eigenvalues are numbered for small α in accordance with (1.4) and (1.5), while the odd spectral numbers correspond to symmetric perturbation modes. For small α the decrements follow the function (1.4) (given in Fig. 2 by a dot−dash line for n = 1), while the phase velocities X_n are close to the average velocity of the flow [see, e.g., Eq. (1.7)]. Subsequently there is a radical rearrangement in the spectrum. The functions $Y_n(\alpha)$ intersect for various spectral numbers. In particular it is interesting to note that for $\alpha \approx 10^{-2}$ the most weakly damped perturbation is the antisymmetric mode (n = 2), while for $\alpha > 1.5$ it is the symmetric mode with n = 3. Nevertheless, the instability of Poiseuille flow is connected only with the first symmetric mode as was assumed in [1].

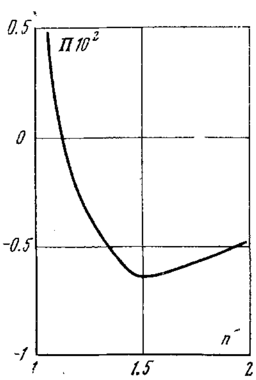

Fig. 4

As α increases the perturbation modes divide into two distinct classes:

a) those close to the wall ($n = 1, 2, 5, 8$ in Figs. 1, 2), where the phase velocity tends to zero (i.e., the velocity of the wall) as α (or R) increases while $y_c \to \pm 1$;

b) those close to the axis ($n = 3, 4, 6, 7$), where the phase velocity tends to the maximum velocity of the fundamental flow and $y_c \to 0$.

When $\alpha \gtrsim R/10$, the quantities $y_c \to 0(\alpha)$ extend to the function (1.4) for all the modes.

The functions $C_n(\alpha)$ in Figs. 1 and 2 give a good illustration of the second locality property (b).

When $\alpha > 2$, the nature of the conditions on the channel axis (symmetry or antisymmetry) ceases to be important for the perturbations close to the wall, and the eigenvalues corresponding to the symmetric and antisymmetric modes merge asymptotically as α increases. The perturbations in the region of the axis also behave characteristically. While the phase velocities are not very large (i.e., y_c is comparatively far removed from the axis), the eigenvalues for the symmetric and antisymmetric modes practically coincide for $\alpha > 0.5$. However, for $\alpha > 10$, when the axis falls in the small region of the critical point, the conditions of symmetry or antisymmetry become important and the corresponding eigenvalues stratify once again.

4. The Effect of a Local Deformation of the Velocity Profile. In order to illustrate the third locality property (c) we investigate the eigenvalue spectrum of the profiles u(y) of the class

$$u = {}^3/_2\,(1 - y^2) + \varepsilon \exp\{-2000\,(y - y_0)^2\}$$

The large factor in the index of the exponential ensures the locality of the deformation, y_0 is the point in the neighborhood of which the deformation occurs, and ε is the amplitude of the deformation.

The functions $Y_n(y_c)$ for the first four spectral numbers are given in Fig. 3 by the solid lines for $\varepsilon = 0.02$, $y_0 = 0$, $R = 10^4$. The dashed lines correspond to the Poiseuille parabola ($\varepsilon = 0$).

In accordance with Fig. 1, small values of α correspond to $y_c \sim 0.2$–0.6, while as α increases the critical points tend to the axis ($y_c \to 0$) and to the wall ($y_c \to 1$).

For modes close to the wall ($n = 1,2$) the functions coincide for $\varepsilon = 0$ and $\varepsilon = 0.02$ within the limits of accuracy of the graphical representation, i.e., a small deformation of the profile towards the axis does not change the eigenvalues of the modes close to the walls, in complete agreement with the locality property (c).

The long-wavelength perturbations are also insensitive to a small deformation of the profile since they depend upon its integral [see, e.g., Eq. (1.7)].

However, the short-wavelength perturbations close to the axis differ radically in these two cases. If the third mode in Poiseuille flow is damped for all Reynolds' numbers, then for a deformed profile it contains increasing perturbations even for $R = 10^4$, as can be seen from Fig. 3. This instability is associated with the appearance of an inflection point in the profile, in accordance with the Tollmin–Rayleigh [1] theorem. Numerical calculations show that as R increases, the critical point of the most rapidly increasing perturbation tends to the inflection point $y = 0.055$. The fourth mode for $R = 10^4$ is damped in both cases, but for the deformed profile with $R > 1.7 \cdot 10^5$ it contains increasing perturbations, while the critical point of the most unstable perturbation tends to the second inflection point $y = 0.016$. In this latter case, the instability is associated with antisymmetric perturbations.

The case in which $\varepsilon = 0.02$, $y_0 = 0.9$ was also calculated, Under these conditions the eigenvalues of the modes in the neighborhood of the axis did not differ from the case $\varepsilon = 0$ to within three significant figures, while for the short-wave perturbations close to the walls they differed significantly. It is characteristic that since the increasing perturbations for Poiseuille flow are comparatively long-wave ($\alpha = 0.9$ for the most unstable perturbation while $R = 10^4$ and decreases as $\sim R^{-1/7}$), a small local deformation occurs in the neighborhood of the critical point. However, in the process, increasing short-wave perturbations appear with a wavelength of the order of the deformation zone.

5. The Most Unstable Perturbations. The fact that instability of the fundamental flow can be associated with several spectral modes of the perturbations (as is clear from Fig. 3, for example) makes the analysis of instability considerably more complicated. It should be noted, however, that it is sufficient to investigate the local maxima of the functions $Y_n(\alpha)$. If $\Pi \equiv \max_\alpha Y < 0$ for all the modes, then stability is certain.

For convex analytic profiles these maxima are divided into three groups (Figs. 1 and 2); long-wave maxima with $\alpha_{max} \lessgtr 1$($n = 1.5$), which are responsible for global stability of the flow, short-wave maxima close to the axis ($n = 3, 4, 6, 7$) and long-wave maxima close to the wall ($n = 1, 2, 5, 8$), which are responsible for the local stability of the zones next to the axis and next to the walls, respectively. As n increases, the critical points of the short-wavelength maxima tend to the point where the local velocity of the stream coincides with the mean velocity along the channel.

In each of these groups there exists an n for which $\max_\alpha Y_n$ lies above the others. In the case under consideration, these are the first and third modes. In what follows those perturbations with wavelengths corresponding to $\max_n \Pi$ will be called the most unstable perturbations.

Since the locality properties enable us to examine global flow stability and the local stability of its individual regions independently, we can treat the most unstable long-wave and short-wave perturbations localized in the axial and the wall zones independently.

The mostunstable short-wave perturbations for symmetric convex profiles u(y) will be localized close to the velocity maximum and close to the channel wall. Let the wavenumber α_{max} of the most unstable perturbation be large enough so that in the region $|y - y_1| \lessgtr 1/\alpha$ (where y_1 is the wall or the channel axis respectively) we can represent the velocity profile in the form

$$u = u(y_1) + \gamma (y - y_1)^n$$

i.e., the remaining terms in the power series expansion of u(y) can be neglected.

Using the locality properties we have, for large enough values of R, after some straight-forward transformations

$$\alpha_{max} = \alpha_* |R\gamma|^{1/n+1} \tag{5.1}$$

$$C(\alpha_{max,\,R}) = u(y_1) + C_*\gamma |R\gamma|^{-n/(n+1)} \tag{5.2}$$

Here α_* and C_* are certain constants associated with the quantity n and independent of R and γ.

Numerical calculations for short-wavelength maxima in the region of the walls and axis were carried out within a wide range of variation of the parameters γ and n for profiles of the type (3.1), and for a large range of variation of the Reynolds' number. The functions (5.1) and (5.2) are well satisfied not only for flows with integral n (Poiseuille and Couette flow, for example), but also for profiles with weak singularities in the region of the velocity maximum.

In particular, in connection with the principle of the maximum instability [3], the following family of profiles was investigated:

$$u = 1 - \gamma y^n, \qquad 1 \leqslant n \leqslant 2$$

For $R > 10^3$ the functions (5.1) and (5.2) were satisfied with a high degree of accuracy for $n > 1.2$. The function $\operatorname{Im} C_* = \Pi(n)$ constructed from numerical calculations for $R = 10^4$ is given in Fig. 4.

Calculations showed that for $R > 10^3$ profiles with $n < 1.12$ are unstable. Profiles with $n < 1.12$ smoothed out in a small neighborhood of $y = 0$ to prevent u'' from going to infinity, were also unstable for large R.

These results contradict those obtained by Potter [4] concerning the stability of triangular and near-triangular velocity profiles. This is explained by the fact that Potter confined his investigations to modes close to the wall, while the instability of the triangular profile is associated with the modes next to the axis.

6. Optimization of the Numerical Method. The locality properties enable the algorithm for numerical calculations of the eigenvalues (1.1) to be optimized for large values of α.

We introduce the functions $A_{ij}(y)$; i, j = 1, 2 such that

$$\varphi = A_{11}\varphi'' + A_{12}\varphi''', \qquad \varphi' = A_{21}\varphi'' + A_{22}\varphi''', \qquad \| A_{ij} \| = A$$

Requiring that φ should satisfy Eq. (1.1), we obtain a system of four nonlinear differential equations for the coefficients of the matrix A. We shall integrate this system from one of the walls with the initial conditions $A_{ij} = 0$; i, j = 1, 2, which ensures that the adherence conditions (1.2) are satisfied for arbitrary values of φ'' and φ'''. If the solution integrated from one of the walls is denoted by a plus superscript and the other by a minus, and we require that φ and its first three derivatives should be continuous at the critical point y_c (or close to it), we obtain the characteristic equation for C:

$$\det(A^+ - A^-)|_{y=y_c} = 0$$

Since the locality properties for large α mean that the eigenvalue is independent of the form of the profile far from y_c and of the nature of uniform boundary conditions on boundaries far from y_c, the integration interval can be restricted to the small neighborhood of y_c and consequently the homogeneous boundary conditions can be transferred to the boundary of this region. This results in a considerable economizing of machine time for large values of α without reducing the accuracy of the eigenvalue calculations.

In summing up, we note that the locality properties formulated above enable us, while investigating the global stability of a given velocity profile, to investigate at the same time, but independently, the stability of individual regions of this profile relative to perturbations of wavelength which do not exceed the dimensions of the region.

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